Decomposing the Higman-Sims graph into double Petersen graphs

Ivana Ilić[1], Nicola Pace[2] and Spyros S. Magliveras[2]
[1] *Math. & Sciences, Edison State College*
Fort Myers, FL 33919, USA
e-mail: iilic@edison.edu
[2] *CCIS, Department of Math. Sciences, Florida Atlantic University,*
Boca Raton, FL 33431, USA
e-mail: npace@fau.edu, spyros@fau.edu

Abstract. It has been known for some time that the Higman-Sims graph can be decomposed into the disjoint union of two Hoffman-Singleton graphs. In this paper we establish that the Higman-Sims graph can be edge decomposed into the disjoint union of 5 double-Petersen graphs, each on 20 vertices. It is shown that in fact this can be achieved in 36960 distinct ways. It is also shown that these different ways fall into a single orbit under the automorphism group HS of the graph.

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Introduction

In 1967, D. G. Higman and C. C. Sims [6] discovered a new sporadic simple group G of order 44,352,000, as a group of automorphisms of a strongly regular graph Γ, the so called *Higman-Sims* graph. Higman and Sims constructed the graph with parameters $(n, k, \lambda, \mu) = (100, 22, 0, 6)$ using the Witt system S(3,6,22) [15] in a clear and elegant way. The group $G = HS$ is a rank-3 primitive extension of the Mathieu group M_{22} and a subgroup of index 2 in the full automorphism group $\overline{G}$ of Γ. In 1978, the third author (S.S.M.) joined the University of Nebraska, Lincoln and was astonished to witness that indeed the Higman-Sims graph Γ had been known by his colleague and friend Dale Mesner some eleven years before it was independently re-discovered by Higman and Sims. Mesner had discovered the graph in his 1956 Ph.D. dissertation as graph $NL_2(10)$ of *negative Latin square type* [12] see also [4]. Suffices to say that Mesner did not consider the automorphism group of $NL_2(10)$. In his 1970 Ph.D. dissertation, S.S.M. determined the maximal subgroups of HS and in the process, discovered that the Petersen graph was contained in Γ [10,11]. In particular if P is the adjacency matrix of the Petersen graph

π, and $z \in G$ is an involution fixing points, then $|\text{fix}\,(z)|$ = 20, and Γ restricted to fix (z) is the "*double Petersen*" graph $\psi := P \otimes J_2$, where J_2 is the 2×2 all-ones matrix. In a recent paper [5] the author briefly discusses that Γ can be decomposed into the disjoint union of graphs isomorphic to ψ. In this article we further investigate decompositions of Γ into double Petersen graphs, present a group theoretic approach to constructing all such decompositions, and prove that all decompositions of Γ into double Petersen graphs fall into a single orbit under the action of G.

1. Preliminaries

Let $\mathcal{G} = (X, E)$ be an undirected graph without loops, where X is the set of vertices and E the set of edges of $\mathcal{G}$. If $x, y \in X$, we denote by $d(x, y)$ the distance in $\mathcal{G}$ between x and y. Further, if r is a non-negative integer, by the *sphere* of radius r about x we mean the set:

$$S_r(x) := \{y \in X \mid d(x, y) = r\}$$

We will denote the Higman-Sims graph by $\Gamma = (X, E)$, where $X = \{0, 1, 2, \ldots, 99\}$ and E is the set of 1100 edges of Γ. Thus, Γ is connected, of diameter 2 and :

1. For $x \in X$, $|S_1(x)| = 22$ and $|S_2(x)| = 77$,

2. If $y \in S_1(x)$, then $|S_1(x) \cap S_1(y)| = 0$, $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 21$ and $|S_2(x) \cap S_2(y)| = 56$,

3. If $y \in S_2(x)$, then $|S_1(x) \cap S_1(y)| = 6$, $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 16$, and $|S_2(x) \cap S_2(y)| = 60$.

4. For $x \in X$, we denote by D_x the 3-(22,6,1) design whose points are the elements of $S_1(x)$ and blocks the 77 subsets $\{B_y = S_1(x) \cap S_1(y) \mid y \in S_2(x)\}$ of X.

If a and b are positive integers and $a < b$ we set $[a, b] = \{a, a+1, \ldots, b-1, b\}$. Further, 0 and 100 denote the same vertex of Γ.

For definitions and elementary properties of group actions the reader is referred to [13,2, 8]. Here, we denote an action of group G on set X by $G|X$. If $x \in X$, $g \in G$, we write x^g for the image of x under g. Moreover, if $A \subseteq X$, we write $G_{[A]}$ for the *pointwise* stabilizer in G of A, and $G_{(A)}$ for the *setwise* stabilizer of A. If $P = \{A_1, A_2, \ldots, A_k\}$ is a partition of X, we write $G_{[[A_1],\ldots,[A_k]]}$ for the subgroup of G which fixes each of the blocks of the partition pointwise, further, we write $G_{[(A_1),\ldots,(A_k)]}$ for the subgroup of G fixing each of the blocks A_i setwise, i.e. possibly permuting the elements within each A_i. Finally, we write $G_{((A_1),\ldots,(A_k))}$ for the subgroup of G fixing the partition as a whole, i.e. which (possibly) permutes the blocks A_i among themselves. Clearly, $G_{[[A_1],\ldots,[A_k]]} \leq G_{[(A_1),\ldots,(A_k)]} \leq G_{((A_1),\ldots,(A_k))}$.

If G is an arbitrary finite group and $x \in G$, we denote by $C(x) = C_G(x)$ the centralizer of x in G, that is, $C(x) := \{y \in G \mid xy = yx\}$. We denote by σ_x the order of $C(x)$. If $K_1 = \{1\}, K_2, \ldots, K_c$, are the conjugacy classes of G, we write σ_i for the order of $C(x_i)$, where $x_i \in K_i$. We write:

$$[K_i \times K_j \to K_k] := \{(a,b) \in K_i \times K_j \mid ab \in K_k\}, \quad i,j,k \in \{1,\ldots,c\} \tag{1}$$

and denote the cardinality of $[K_i \times K_j \to K_k]$ by $|K_i \times K_j \to K_k|$. Further, we write:

$$\langle K_i \times K_j \to K_k \rangle = \{\langle a,b \rangle \mid (a,b) \in [K_i \times K_j \to K_k]\} \tag{2}$$

where $\langle a,b \rangle$ denotes the subgroup of G generated by a and b.

The structure constants of the center of the group algebra $\mathbb{Z}G$ are denoted by $a_{i,j,k}$, thus,

$$K_i K_j = \sum_{k=1}^{c} a_{i,j,k} K_k \quad i,j \in \{1,\ldots,c\}; \tag{3}$$

we also have:

$$a_{i,j,k} = \frac{|G|}{\sigma_i \sigma_j} \sum_{t=1}^{c} \frac{\chi_t(i)\chi_t(j)\overline{\chi_t(k)}}{\chi_t(1)} \tag{4}$$

where $\chi_t(i)$ is the value of the irreducible ordinary character χ_t of G on the elements of the class K_i. The character table of G is presented below.

Character Table of HS:

x	1	2_1	2_2	4_1	4_2	4_3	8_1	8_2	8_3	3	6_1	6_2	12	5_1	5_2	5_3	10_1	10_2	20_+	20_-	15	7	11	11
σ_x	$\lvert G \rvert$	7680	2880	64	256	3840	16	16	16	360	24	36	12	25	300	500	20	20	20	20	15	7	11	11
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	−2	2	2	−6	0	0	0	4	0	−2	0	2	2	−3	−2	1	−1	−1	−1	1	0	0
χ_3	77	13	1	1	5	5	1	−1	−1	5	1	1	−1	2	−3	2	1	−2	0	0	0	0	0	0
χ_4	175	15	11	3	−1	15	−1	1	1	4	0	2	0	0	5	0	1	0	0	0	−1	0	−1	−1
χ_5	154_a	10	10	−2	6	−2	0	0	0	1	1	1	1	−1	4	4	0	0	−2	−2	1	0	0	0
χ_6	154_b	10	−10	2	−2	−10	0	2	−2	1	1	−1	−1	−1	4	4	0	0	0	0	1	0	0	0
χ_7	154_c	10	−10	2	−2	−10	0	−2	2	1	1	−1	−1	−1	4	4	0	0	0	0	1	0	0	0
χ_8	231	7	−9	−1	−1	15	−1	−1	−1	6	−2	0	0	1	1	6	1	2	0	0	1	0	0	0
χ_9	693	21	9	1	5	21	1	−1	−1	0	0	0	0	−2	3	−7	−1	1	1	1	0	0	0	0
χ_{10}	770_a	34	−10	−2	2	−14	−2	0	0	5	1	−1	1	0	0	−5	0	−1	1	1	0	0	0	0
χ_{11}	770_b	−14	10	−2	−2	−10	0	0	0	5	1	1	−1	0	0	−5	0	1	θ	$\overline{\theta}$	0	0	0	0
χ_{12}	770_c	−14	10	−2	−2	−10	0	0	0	5	1	1	−1	0	0	−5	0	1	$\overline{\theta}$	θ	0	0	0	0
χ_{13}	825	25	9	1	1	−15	1	1	1	6	−2	0	0	0	−5	0	−1	0	0	0	1	−1	0	0
χ_{14}	896_a	0	16	0	0	0	0	0	0	−4	0	−2	0	1	1	−4	1	0	0	0	1	0	ϕ	$\overline{\phi}$
χ_{15}	896_b	0	16	0	0	0	0	0	0	−4	0	−2	0	1	1	−4	1	0	0	0	1	0	$\overline{\phi}$	ϕ
χ_{16}	1056	32	0	0	0	0	0	0	0	−6	2	0	0	1	−4	6	0	2	0	0	−1	−1	0	0
χ_{17}	1386	−6	18	−2	−2	6	0	0	0	0	0	0	0	1	6	11	−2	−1	1	1	0	0	0	0
χ_{18}	1408	0	16	0	0	0	0	0	0	4	0	−2	0	−2	−7	8	1	0	0	0	−1	1	0	0
χ_{19}	1750	−10	10	2	6	−10	−2	0	0	−5	−1	1	−1	0	0	0	0	0	0	0	0	0	1	1
χ_{20}	1925_a	5	−19	−3	5	5	1	1	1	−1	−1	−1	−1	0	5	0	1	0	0	0	−1	0	0	0
χ_{21}	1925_b	5	1	1	−3	−35	1	−1	−1	−1	−1	1	1	0	5	0	1	0	0	0	−1	0	0	0
χ_{22}	2520	24	0	0	−8	24	0	0	0	0	0	0	0	0	0	−5	0	−1	−1	−1	0	0	1	1
χ_{23}	2750	−50	−10	2	2	10	0	0	0	5	1	−1	1	0	0	0	0	0	0	0	0	−1	0	0
χ_{24}	3200	0	−16	0	0	0	0	0	0	−4	0	2	0	0	−5	0	−1	0	0	0	1	1	−1	−1

2. Petersen subgraphs in Γ

To present various aspects of the problem in a compact way, we begin by specifying generators α and β of G as permutations of degree 100. Then, strong generators for G can be computed from α and β using the Schreier-Sims algorithm (or variation) as found in a computer algebra system like MAGMA or GAP. The permutations α, β have been chosen in a canonical way, to demonstrate directly the nature of the objects under consideration. To conserve space we have written α and β as :

$$\alpha = \left(\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 38 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 2 & 20 & 81 & 82 & 83 & 84 & 86 & 85 & 88 & 87 & 91 \\ 3 & 92 & 90 & 89 & 96 & 95 & 93 & 94 & 99 & 0 & 98 \\ 4 & 97 & 70 & 69 & 79 & 80 & 64 & 63 & 76 & 75 & 74 \\ 5 & 73 & 68 & 67 & 61 & 62 & 71 & 72 & 77 & 78 & 65 \\ 6 & 66 & 53 & 54 & 46 & 45 & 59 & 60 & 52 & 51 & 42 \\ 7 & 41 & 55 & 56 & 50 & 49 & 48 & 47 & 57 & 58 & 43 \\ 8 & 44 & 21 & 22 & 23 & 24 & 26 & 25 & 28 & 27 & 32 \\ 9 & 31 & 29 & 30 & 35 & 36 & 34 & 33 & 40 & 39 & 37 \end{array}\right), \quad \beta = \left(\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 1 & 59 & 64 & 16 & 32 & 72 & 91 & 69 & 81 & 17 \\ 1 & 92 & 90 & 9 & 74 & 19 & 93 & 97 & 36 & 41 & 79 \\ 2 & 8 & 14 & 71 & 98 & 78 & 21 & 44 & 39 & 63 & 68 \\ 3 & 87 & 96 & 54 & 24 & 4 & 26 & 82 & 86 & 34 & 62 \\ 4 & 30 & 47 & 84 & 5 & 75 & 20 & 27 & 42 & 57 & 15 \\ 5 & 2 & 53 & 28 & 40 & 38 & 50 & 73 & 11 & 88 & 6 \\ 6 & 48 & 55 & 67 & 10 & 61 & 80 & 13 & 46 & 49 & 83 \\ 7 & 35 & 66 & 23 & 99 & 22 & 70 & 37 & 76 & 31 & 25 \\ 8 & 56 & 85 & 12 & 94 & 18 & 45 & 95 & 51 & 3 & 7 \\ 9 & 60 & 0 & 52 & 29 & 89 & 77 & 33 & 58 & 43 & 65 \end{array}\right)$$

Reading α and β is straight forward. For example, $\alpha(83) = 23$ and $\beta(8) = 81$. We easily recover the graph Γ by first computing the stabilizer G_x for any particular $x \in X$, then computing the point orbits of G_x on X. These orbits will have lengths 1, 22 and 77. We select the orbit of length 22 as the set of neighbors of x. In particular for $0 \in X$ and the generators α and β given,

$$S_1(0) = \{4, 6, 18, 20, 22, 29, 34, 37, 43, 46, 58, 59, 61, 71, 75, 79, 87, 88, 89, 90, 93, 94\}$$

Because $S_1(x^g) = (S_1(x))^g$, for any $x \in X,\ g \in G$, by the transitivity of G on X we easily obtain the spheres of radius 1 with centers at each vertex of the graph.

The Higman-Sims group has 24 conjugacy classes of elements. In particular there are two conjugacy classes of involutions, denoted here by K_{2_1} and K_{2_2}. Further, there is a single class of elements of order 3, denoted by K_3, and there are 3 conjugacy classes of elements of order 5, which we denote by K_{5_1}, K_{5_2} and K_{5_3}. If $z \in K_{2_1}$, then z fixes exactly 20 points of X, on the other hand, elements of K_{2_2} fix no points. The three conjugacy classes of elements of order 5, are characterized by the orders of their centralizers. If f is an element of order 5 in G, then $|C(f)| = 25$, 300, or 500 according to whether $f \in K_{5_1}, K_{5_2}$ or K_{5_3}.

The following proposition deals with the existence of Petersen and double Petersen graphs in Γ and characterizes such subgraphs.

Proposition 2.1 *(i) Let $z \in G$ be an involution that fixes points of X, then the subgraph of Γ with vertex set* fix (z) *is isomorphic to the double Petersen graph ψ.*
(ii) If γ is a Petersen subgraph of Γ with vertex set Y then the pointwise stabilizer of Y is of order 2, in particular, $G_{[Y]} = \langle z \rangle$, where $z \in K_{2_1}$, and γ is a subgraph of a double Petersen graph.

Proof. It was already observed in [11] that if $z \in G$ is an involution which fixes points of X, i.e. $z \in K_{2_1}$, then the subgraph with vertex set fix (z) constitutes a double Petersen graph. Here, we give a proof of this fact by examining the generator α of G. We see directly that the 20 points $\{1, 2, \ldots, 20\}$ fixed by α constitute a subgraph of Γ isomorphic to the double Petersen graph ψ.

Now, suppose that γ is any Petersen subgraph of Γ with vertex set $Y = \{y_0, \ldots, y_9\}$, and label the y_i so that y_0 is adjacent to $\{y_1, y_2, y_3\}$. We proceed to determine $G_{[Y]}$. Because G is transitive on X we have that $|G_{y_0}| = |G|/100 = 443520$. We also know that G_{y_0} is isomorphic to M_{22} and is 3-transitive on $S_1(y_0)$. Thus, $G_{[y_0,y_1]}$ is isomorphic to $M_{21} \cong PSL_3(4)$, has order 20160 = 443520/22 and is 2-transitive on $S_1(y_0) \setminus \{y_1\}$. Because $y_2 \in S_1(y_0) \setminus \{y_1\}$, we have that $G_{[y_0,y_1,y_2]}$ is isomorphic to M_{20}, has order 960 = 20160/21, and is transitive on the 20 points of $S_1(y_0) \setminus \{y_1, y_2\}$. But $y_3 \in S_1(y_0) \setminus \{y_1, y_2\}$, so we have that $H = G_{[y_0,y_1,y_2,y_3]}$ has order 48 = 960/20. Now, relabel $\{y_4, \ldots, y_9\}$ as $\{y_{i,j} \mid 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$, so that for $i \in \{1, 2, 3\}$ $S_1(y_i) \cap Y = \{y_0, y_{i,1}, y_{i,2}\}$. The 6 elements $\{y_{i,j}\}$ are no longer in $S_1(y_0)$, but we now find that for any particular $i \in \{1, 2, 3\}$ $G_{[y_0,y_1,y_2,y_3,y_{i,1}]} = H_{y_{i,1}}$, has order 4 and fixes point $y_{i,2}$ as well. Finally, computing $K = H_{[y_{1,1},y_{2,1}]}$ we see that K is of order 2 and fixes pointwise all of Y. Then $K = \langle z \rangle$ for some involution of G, and because z fixes points, we have that $z \in K_{2_1}$. Because $|\text{fix}\,(z)| = 20$, we let $\overline{Y} = \text{fix}\,(z) \setminus Y$. Since we already know that all involutions fixing points are conjugate in G, and since we have already noted that the subgraph on the fix of one involution in K_{2_1} is a double Petersen graph, the same will be true for the subgraph on fix (z). □

Corollary 2.1 *The Higman-Sims graph Γ contains subgraphs isomorphic to the double Petersen graph ψ, and consequently Γ also contains Petersen subgraphs.*

Proposition 2.2 *If $z \in K_{2_1}$ and $A = \text{fix}\,(z)$, then $G_{(A)} = C_G(z)$, is of order 7680, is transitive on A and on $X \setminus A$, and is a maximal subgroup of G.*

Proof. The result follows directly from the list of maximal subgroups of G and their structure as stated in [11]. □

Let $\mathcal{B} = \{A \subset X \mid A = \text{fix}\,(z), z \in K_{2_1}\}$, then for any $z \in K_{2_1}$, $|\mathcal{B}| = [G : C(z)] = 5775$.

Proposition 2.3 *Let $B \in \mathcal{B}$, then, for each $x \in B$ there is a uniquely determined element $x' \in B$ such that $x \neq x'$ and $x'' = x$. Thus, B can be written as $\{\{x_1, x_1'\}, \ldots, \{x_{10}, x_{10}'\}\}$.*

Proof. Suppose $B \in \mathcal{B}$, and let $z \in K_{2_1}$ be the involution with fix $(z) = B$. For $x \in B$, note that $z \in G_x \cong M_{22}$, and that G_x acts as a group of automorphisms of the 3-(22,6,1) design D_x, with points $S_1(x)$ and blocks indexed by $S_2(x)$. It is known that in the 3-transitive representation of M_{22} on 22 points, an involution fixes exactly 6 points, thus, $|S_1(x) \cap B| = 6$. In particular, the six points fixed by an involution in M_{22} constitute a (design) block in D_x. Thus, z fixes a particular point $x' \in S_2(x)$ corresponding to the design block of these 6 points. □

Remark 2.1 Clearly the *pairing* $x \leftrightarrow x'$ is strictly dependent on B. Further we note that for $x \in B \in \mathcal{B}$ the proof in Proposition 2.3 accounts for 8 points of $\text{fix}\,(z)$, namely $\{x, x'\} \cup (S_1(x) \cap B) = \{x, x'\} \cup (S_1(x) \cap S_1(x'))$. The structure of the remaining 12 points of $\text{fix}\,(z)$ is easily seen to be as follows. Let $\{x_1, x_1'\}, \{x_2, x_2'\}, \{x_3, x_3'\}$ be the pairing induced by B on $S_1(x) \cap B$. For each $i \in \{1,2,3\}$, $S_1(x_i) \cap S_1(x_i')$ consists of 6 points, two of which are x and x'. The 12 remaining points of B are $\cup_{i=1}^{3}(S_1(x_i) \cap S_1(x_i')) \setminus \{x, x'\}$.

3. Decompositions

We now wish to examine whether it is possible to find a partition $\{B_i\}_{i=1}^{5}$ of X consisting of blocks $B_i \in \mathcal{B}$. If such a partition exists the induced subgraphs on the blocks of the partition would constitute an edge decomposition of Γ into double Petersen graphs. In this section we show that such partitions exist and they all fall into a single orbit under the action of G. Moreover, we are able to count the total number of distinct such partitions.

Let A be a 5775×100 (0,1)-matrix whose rows are the characteristic vectors of the distinct sets B_i in $\mathcal{B}$. In particular, arrange A so that its i^{th} row is the fix of the i^{th} involution in K_{2_1}. We see A as the incidence matrix of a combinatorial design $\mathcal{D} = (X, \mathcal{B})$ with points X and blocks $\mathcal{B}$. $\mathcal{D}$ is a 1-(100,20,1155) design and fails to be a 2-design. If $y \in S_1(x)$ then there are $\lambda_1 = 315$ blocks containing $x, y \in X$, while if $y \in S_2(x)$ there are $\lambda_2 = 195$ blocks passing through x and y.

We define a graph with vertices the blocks in $\mathcal{B}$, where two blocks are adjacent if and only if they are disjoint. We then search for 5-cliques in this graph. Our program SYNTH, which was designed to construct $t-(v,k,\lambda)$ designs from Kramer-Mesner matrices [9], determines all possible solutions from A in less than 2 hours on a desktop computer. There are in all 36960 cliques of size 5. We consider the first solution, and collect the corresponding 5 involutions $z_1', z_2', \ldots, z_5'$ for this solution. We subsequently relabel Γ by conjugating simultaneously the 5 involutions and initial generators of G by a sequence of transpositions of the symmetric group $\mathcal{S}_{100}$ so that, after relabeling, the five involutions become $z_1 = \alpha, z_2, \ldots, z_5$, with $\text{fix}\,(z_i) = \{20(i-1) + j \mid j \in \{1, 2, \ldots, 20\}\}$.

Theorem 3.1 *(i) There exists an edge decomposition* $\rho = \{B_i\}_{i=1}^{5}$ *of* Γ *into the disjoint union of 5 double Petersen graphs.*

(ii) There are in all 36960 distinct partitions of X *into 5 disjoint blocks* $\{B_i\}_{i=1}^{5}$ *with* $B_i \in \mathcal{B}$*, i.e. 36960 decompositions of* Γ *into double Petersen graphs.*

(iii) If z_1, z_2 *correspond to the blocks* B_1 *and* B_2 *above, then* $\langle z_1, z_2\rangle$ *is a dihedral subgroup of order 10, denoted by* D_5*, and belongs to* $\langle K_{2_1} \times K_{2_1} \to K_{5_2}\rangle$*, further* $z_3, z_4, z_5 \in D_5$.

(iv) The cyclic subgroup of order 5 in D_5 *permutes the blocks of* ρ *among themselves, and the centralizer* $C_G(D_5)$ *is a subgroup of* G *isomorphic to the alternating group* A_5.

Proof. (i) and (ii) We seek solutions $X \in \{0,1\}^{5775}$ to the matrix system of equations

$$XA = J$$

where J is the 1×100 row vector. There are in all 36960 solutions. To exhibit a first solution consider the subgroup $D_5 = \langle \alpha, z_2 \rangle$ where α is the generator of G mentioned in Section 2, and $z_2 = (\alpha\beta^2\alpha\beta\alpha\beta^2\alpha\beta^4\alpha\beta^4\alpha\beta^2)^4$. The subgroup D_5 is dihedral of order 10, and its 5 involutions $z_1 = \alpha, z_2, \ldots, z_5$ fix respectively (and pointwise) the five blocks $B_1 = [1, 20]$, $B_2 = [21, 40]$, $B_3 = [41, 60]$, $B_4 = [61, 80]$, $B_5 = [81, 99] \cup \{0\}$. Thus, $z_1, \ldots, z_5$ are in K_{2_1}. We further verify that an element of order 5 in D_5 permutes the blocks B_i among themselves, thus, $D_5 = G_{([B_1],\ldots,[B_5])}$.
(iv) Computing $C := C_G(D_5) = C_G(z_1) \cap C_G(z_2)$ yields that $|C| = 60$. Further investigation yields that there are two elements $\delta, \tau \in C$ such that δ is an involution fixing no points, τ is an element of order 3, and $\delta\tau$ is of order 5 fixing no points. Thus, C is isomorphic to the alternating group A_5. We further verify that C is transitive on each of the five blocks B_i, that is $C \subseteq G_{[(B_1),\ldots,(B_5)]}$. It follows that there is a subgroup of G which is the direct product $D_5 \times C$ of order 600. Now, because an element f of order 5 in D_5 commutes with an element of order 3 in C, all the elements of order 5 in D_5 come from K_{5_2}. Further, we find that the order of the centralizer of an element g of order 5 in C is 500, thus all elements of order 5 in C come from K_{5_3}, i.e., $C \in \langle K_{2_2} \times \mathcal{K}_3 \to K_{5_3} \rangle$. We finally note that $D_5 \times C$ is contained in the normalizer of C (as well as the normalizer of D_5). Computing the normalizer $N = N_G(C)$ we find that N is of order 1200, and is transitive on X. Moreover checking with the list of maximal subgroups of G we see that indeed $N_G(C) = N_G(D_5)$ is a maximal subgroup of G, thus, $N = G_{((B_1),\ldots,(B_5))}$. But $[G : N] = 44352000/1200 = 36960$, Thus, the 36960 decompositions constitute a single orbit under G.

Because there is a one-to-one correspondence between edge decompositions of Γ into double Petersen graphs and dihedral groups of type $(2_1, 2_1, 5_2)$, a second way of counting decompositions is to count all dihedral subgroups in $\langle K_{2_1} \times K_{2_1} \to K_{5_2} \rangle$. From the character table of G, we compute:

$$a_{2_1,2_1,5_2} = \frac{|G|}{\sigma_{2_1}\sigma_{2_1}} \sum_{t=1}^{24} \frac{\chi_t(2_1)\chi_t(2_1)\overline{\chi_t(5_2)}}{\chi_t(1)} = 5$$

Thus,

$$|K_{2_1} \times K_{2_1} \to K_{5_2}| = 5 \cdot \frac{|G|}{300} = 739200$$

However, in any dihedral D_5

$$|K_2 \times K_2 \to K_{5_+} \cup K_{5_-}| = 20$$

Therefore the total number of D_5's of type $(2_1, 2_1, 5_2)$ is 739200/20 = 36960.

A third, independent way of checking the correctness of the number of decompositions is to count all subgroups isomorphic to A_5 in $\langle K_{2_2} \times K_3 \to K_{5_3} \rangle$. From the character table of G, we compute:

$$a_{2_2,3,5_3} = \frac{|G|}{\sigma_{2_2}\sigma_3} \sum_{t=1}^{24} \frac{\chi_t(2_2)\chi_t(3)\overline{\chi_t(5_3)}}{\chi_t(1)} = 50$$

Thus,

$$|K_{2_2} \times K_3 \to K_{5_3}| = 50 \cdot \frac{|G|}{500} = 4435200$$

However, in any A_5

$$|K_2 \times K_3 \to K_{5_1} \cup K_{5_2}| = 120$$

Therefore the total number of A_5's of type $(2_2, 3, 5_3)$ is 44352/120 = 36960. The centralizer of each such an A_5 is a dihedral subgroup of order 10 and type $(2_1, 2_1, 5_2)$ and each of these dihedrals gives rise to an edge decomposition. □

Dedication

This article is dedicated to the memory of Professor Ralph G. Stanton who passed away on April 21, 2010. Ralph's impact on mathematics and computer science has been most remarkable and was expressed in so many ways that would be impossible to account for here. This paper's oldest author has known Ralph since 1978 having met him for the first time at the Southeastern Combinatorics Conference, in Boca Raton. The two young authors met him for the first time in 2006, at the same Conference. Ralph was one of the founders of the Conference, and to our knowledge, he never missed coming and participating in it since its inception 42 years ago. Ralph's Ph.D. dissertation under Richard Brauer was the characterization of the two 5-transitive Mathieu groups M_{12} and M_{24} by their orders and uses ordinary and modular character theory. It is in the spirit of Ralph's early work, but also his later interests in the sort of combinatorics that deal with problems like the one addressed here, that we present this paper, using a bit of character theory as well. We will never forget Ralph Stanton and his accomplishments as a mathematician, talented planner and executor of great scientific programs, as a friend, and a great human being.

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